METRIC AND TOPOLOGICAL SPACES: EXAM 2020/21

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Problem 1 (10 + 10%). (*a*) Let $\varepsilon > \varepsilon' > 0$. In a metric space $(\mathcal{X}, \mathrm{d}_{\mathcal{X}})$, can an open disk B_ε of larger radius ε be entirely and strictly contained, $B_\varepsilon \subsetneq B_{\varepsilon'}$, in an open disk $B_{\varepsilon'}$ of smaller radius ε' ? (state and prove, e.g., by example)
(*b*) If there *can be* (*a*), then can the diameter of that disk B_ε of larger radius ε be greater than the diameter of the disk $B_{\varepsilon'}$ of smaller radius ε' ?

Problem 2 (20%). Are the subsets $[0,1)\times[0,1)$ and $[0,1]\times[0,1)$ of Euclidean plane $\mathbb{E}^2$ homeomorphic or not ?

(Either prove the impossibility of any homeomorphism $\sim$ or describe one homeomorphism $\sim$ explicitly.)

Problem 3 (20%). Let $(\mathcal{X}, \mathrm{d}_{\mathcal{X}})$ be a metric space and $\{U_i \mid i \in \mathcal{I}\}$ be a family of connected subsets $U_i \subseteq \mathcal{X}$ such that $U_i \cap U_j \neq \varnothing$ for all $i, j \in \mathcal{I}$. Prove that the union $U = \bigcup_{i\in\mathcal{I}} U_i$ is connected.

Problem 4 (20%). Suppose for every $n \in \mathbb{N}$ that V_n is a non-empty closed subset of a compact space $\mathcal{X}$ with $V_n \supseteq V_{n+1}$. Prove

$$\bigcap_{n=1}^{+\infty} V_n \neq \varnothing.$$

Problem 5 (20%). Let $(\mathcal{X}, \mathrm{d}_{\mathcal{X}})$ be a non-empty complete metric space and $f, g\colon \mathcal{X} \to \mathcal{X}$ two Banach contractions of $\mathcal{X}$. Does there always exist a point $x_0 \in \mathcal{X}$ such that $f(g(x_0)) = x_0$?

(state and prove)

Date: November 5, 2020 (19:30–21:30). Good luck & take care !